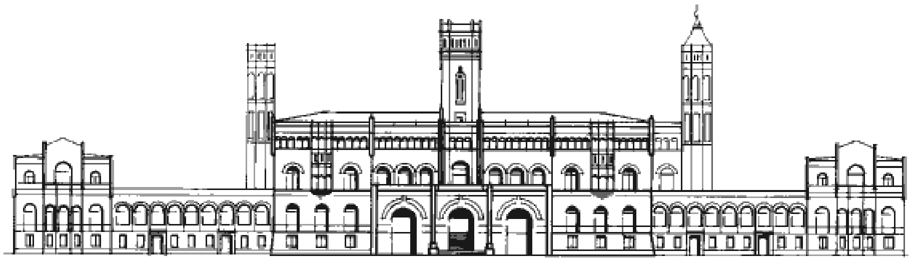
Welfen Laboratory Reports
Editor: Franz–Erich Wolter

Welfen Laboratory, Report No. 1

Computation of Medial Curves on Surfaces

Thomas Rausch **Franz–Erich Wolter** **Oliver Sniehotta**

August 1996



Lehrstuhl Graphische
Datenverarbeitung
Welfenlabor
Institut für Informatik
Universität Hannover
Welfengarten 1
30167 Hannover, Germany

Division of Computer Graphics
and Geometric Modelling
Welfen Laboratory
Institute of Computer Science
University of Hannover
Welfengarten 1
30167 Hannover, Germany

few@informatik.uni-hannover.de
http://www-c.informatik.uni-hannover.de

Welfen Laboratory Reports No. 1

Franz–Erich Wolter (Editor)

Lehrstuhl Graphische Datenverarbeitung
Institut für Informatik
Universität Hannover
Welfengarten 1
30167 Hannover

3. Auflage
1. Februar 2000
(100 Stück)

Computation of Medial Curves on Surfaces

Thomas Rausch Franz–Erich Wolter Oliver Sniehotta

August 1996

Abstract

Medial curves considered in this paper may be regarded roughly spoken as the local equidistantial set of certain subarcs of two given border curves. In contrast to the problem of computing the equidistantial set (which is a *global* problem) we focus our attention on the related *local* problem by ignoring the question whether the computed curve consists only of points that are truly equidistantial to both border curves, the latter ones considered in their full range (cf. figure 6). In this paper a method for tracing the medial curve of two border curves is presented. This method is based on the numerical solution of a system of differential equations in the parameter space of the considered surface. It is proven that this system remains regular as long as the medial curve stays away from focal points of the border curves. Finally, it is indicated how the concept might be extended in order to trace the focal curve of a given curve on the surface. The techniques here are designed to work on arbitrary regular surfaces where distance has to be understood as geodesic distance. The method however is useful in the planar case too.

Keywords: CAD, geometric modeling, equidistantial set, medial curve, medial axis, cut locus, focal curves.

1 Introduction and Related Works

Given two point sets A, B in the Euclidean plane it is very natural to ask how to compute the locus of points being equidistantial to both sets A and B. To solve this problem for say general compact sets is certainly very difficult. Therefore it appears reasonable to start with a relevant special case where the two sets A, B are given by parametrized regular simple differentiable curves, say $A = \{A(t) \mid t \in I\}$, $B = \{B(r) \mid r \in J\}$. Even solving this more special problem in general is not easy at all because it requires to treat a global problem, namely to find the set of points $M(A, B) = \{x \in \mathbb{E}^2 \mid d(A, x) = d(B, x)\}$ where the functions $d(A, x)$, $d(B, x)$ describe the distance of a variable point $x \in \mathbb{R}^2$ to the sets A, B respectively. One has to bear in mind that the global nature of the function $d(A, x)$ is illustrated by the fact that small changes in x may imply a discontinuous change in the distance minimal segment from x to A (see figure 1). Hence the nearest footpoint on the set A to the moving point x may change discontinously with x. Therefore the difficulties of finding the points nearest on A to the moving point x are strongly influenced by the global geometry of A. However in order to compute the function $d(A, x)$ we need to solve the global minimization problem, i.e. to find the point nearest on A to the space point x. Solving this global problem is related to computing the cut locus of a set A. It can be shown that a segment emanating normal from a curve (or surface) A will be distance minimal until it meets the cut locus of A, cf. figure 1). This geometric property can also be

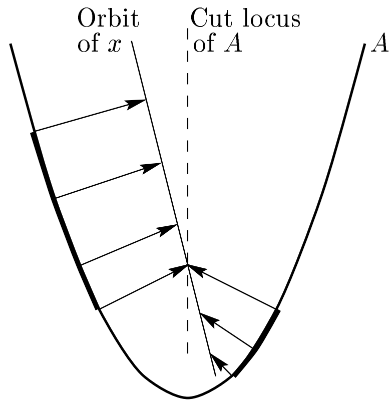


Figure 1: For $x \in$ Cut Locus of A the distance minimal segment from x to A changes discontinously.

viewed as a possible definition for the cut locus C_A relative to a given reference set A where from the segments emanate normally (see [Wol85], [Wol92]).

In this paper we do not want to discuss the global problem computing the equidistantial set of two given curves. We rather want to discuss methods contributing to treat a local problem finding the medial curve of two subarcs of the curves A and B. Even here we will make further restrictions. In this paper we shall not consider the situations where the medial curve contains focal points respective its foot points on the curves A or B. Also we will not discuss the (global) question if the computed (candidate) „medial curve“ contains only points that are truly equidistantial with respect to the considered subarcs of A and B (see figure 6 on page 17 for an example). The focus in this paper is on analytical tools and the distance functions used to compute the equidistantial curve are related to two given families of normal segments (emanating from the arcs A, B respectively) and we assume that those normal segments are distance minimal to the respective subarcs of A and B. We are essentially presenting a differential equation tracing the medial arc by following its tangent vector. The medial arc's tangent vector must bisect the angle of equilengthial normal segments stemming from the two families of normal segments supposed to be distance minimal to the respective sub arcs of A and B. This concept has been considered theoretically in [Wol85], p. 171. The computational aspect of the concept in the Euclidean case had been presented in public in a sketchy way by F.-E. Wolter on a conference in Tempe [Wol89]. In the Euclidean case F.-E. Wolter's algorithm was implemented and tested by E. Sherbrooke at the MIT Design Laboratory in 1989.

However the main contribution in this paper is that goes beyond the Euclidean case and presents analytical methods solving the aforementioned (restricted) computation problem determining medial curves on a surface for two given curves on the surface. The distance is here now understood as the geodesic distance. The main contribution is to present a differential equation useful to trace the medial curve by following its tangent expressed in the differential equation. Obviously the situation on the surface is far more complicated than in the Euclidean plane case. Nonetheless the medial curves tangent must still bisect the angle built by the initial vectors of the two (distance) minimal geodesics connecting the medial curve point with the respective surface curve arcs, see [Wol85], p. 171. In the general surface situation the two families of Euclidean normal segments emanating from the subarcs of A and B are replaced by two families of geodesics emanating orthogonally from the respective subarcs of A and B.

Some key ideas in this paper had been already presented by F.-E. Wolter in seminars in Oberwolfach [Wol95] and Dagstuhl [Wol96]. However these ideas have not yet appeared in print and it was only until recently that the joint effort of the authors of this paper lead to showing that the computational methods described in this paper give indeed practically robust techniques of high accuracy. It is quite useful to view at a medial curve respective two given curves A, B as being obtained via intersection points of offset curves with the same offset distance from the respective progenitor curves A, B. In this context we refer also to [PB89] which studies the computation of geodesic offsets on surfaces. Meanwhile there exist many papers studying medial curves and surfaces in the Euclidean case. We mention here only a few [Gur89], [PG90], [Wol92], [She95], [DH93], [HV94], [BJA91], [SPB95], [SPW95]. The latter contains also an extended up to date bibliography. Quite recently T. Maekawa developed a method to compute geodesics joining two surface points, see [Mae96].

Deriving the computational methods in this papers requires some tools from differential geometry. Some of these tools are classical methods within the area of Riemannian geometry they may however not be standard in the area of geometric modeling. Because of this and in order to present the derivations in a complete and self contained way we include in this paper some background material on tools from differential geometry. Some of this background material is also needed to introduce in a precise way the definitions, notations and concepts used for our computational methods.

2 Background

2.1 Elementary Differential Geometry

A general parametric surface S can be defined by a vector-valued mapping from the two-dimensional parametric space to a set of three-dimensional coordinates

$$\boldsymbol{r}(u,v) \;=\; (x(u,v), y(u,v), z(u,v))^T \,.$$

The mapping $\boldsymbol{r}$ is called a parametrization of the surface S. If for every surface point $\boldsymbol{r}(u_0,v_0) = \boldsymbol{p}$ there exists a neighbourhood $\boldsymbol{p} \in V \subset \mathbb{R}^3$ and $(u_0,v_0)^T \in U \subset \mathbb{R}^2$ such that $\boldsymbol{r} : U \to V \cap S$ is a differentiable homeomorphism on U and the differential $d\boldsymbol{r}_{\boldsymbol{q}} : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one for every point $\boldsymbol{q} \in U$, the surface S itself is called a regular parametric surface.

Any curve $\boldsymbol{\alpha}(t)$ on the surface S can be represented by a curve $(u(t), v(t))$ in the parameter space of S, i.e.

$$\boldsymbol{\alpha}(t) \;=\; \boldsymbol{r}(u(t), v(t)).$$

For any parameter t_0 the tangent vector $\boldsymbol{\alpha}'(t_0)$ can be regarded as a direction in the surface's tangent plane at the point $\boldsymbol{\alpha}(t_0) \in S$. On the other hand: for every vector $\boldsymbol{w}$ of the tangent plane $T_{\boldsymbol{p}}(S)$ in $\boldsymbol{p} \in S$ there exists a curve $\boldsymbol{\alpha}$ on S such that $\boldsymbol{\alpha}(0) = \boldsymbol{p}$ and $\boldsymbol{\alpha}'(0) = \boldsymbol{w}$. One may say that the tangent plane $T_{\boldsymbol{p}}(S)$ at a certain point $\boldsymbol{p}$ is the set of the tangent vectors of all curves on S passing $\boldsymbol{p}$. Moreover, those tangent vectors are given by

$$\boldsymbol{\alpha}'(0) \;=\; u'(0)\,\boldsymbol{r}_u(u(0), v(0)) + v'(0)\,\boldsymbol{r}_v(u(0), v(0)),$$

where $\boldsymbol{r}_u, \boldsymbol{r}_v$ denote the partial derivatives of the parametrization $\boldsymbol{r}$ of S. Hence $T_{\boldsymbol{p}}(S)$ is a two-dimensional vector space spanned by the local coordinate vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$.

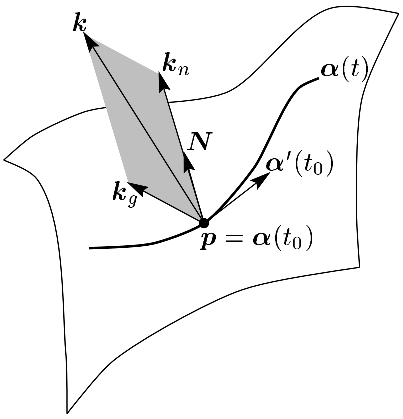


Figure 2: Normal and geodesic curvature vector of a curve $\boldsymbol{\alpha}(t)$ on a surface S.

Many important local geometric entities of the surface can be defined in terms of its first and second fundamental forms. Both are quadratic forms on the tangent planes $T_{\boldsymbol{p}}(S)$. If we take $\boldsymbol{r}_u, \boldsymbol{r}_v$ as the local coordinate system, the representation of the first fundamental form I is given by

$$I(u', v') \;=\; E\,(u')^2 + 2\,F\,u'\,v' + G\,(v')^2$$

where

$$E \;:=\; \langle \boldsymbol{r}_u, \boldsymbol{r}_u \rangle, \quad F \;:=\; \langle \boldsymbol{r}_u, \boldsymbol{r}_v \rangle, \quad G \;:=\; \langle \boldsymbol{r}_v, \boldsymbol{r}_v \rangle$$

and $\langle ., . \rangle$ denotes the inner product. The second fundamental form II is given by

$$II(u', v') \;=\; L\,(u')^2 + 2\,M\,u'\,v' + N\,(v')^2$$

where

$$L \;:=\; \langle \boldsymbol{N}, \boldsymbol{r}_{uu} \rangle, \quad M \;:=\; \langle \boldsymbol{N}, \boldsymbol{r}_{uv} \rangle, \quad N \;:=\; \langle \boldsymbol{N}, \boldsymbol{r}_{vv} \rangle, \quad \boldsymbol{N} \;:=\; \frac{\boldsymbol{r}_u \wedge \boldsymbol{r}_v}{\|\boldsymbol{r}_u \wedge \boldsymbol{r}_v\|}.$$

Here $\wedge$ denotes the cross product of $\mathbb{R}^3$. The vector $\boldsymbol{N}$ is called the normal of the surface S.

In order to quantify the curvature of a surface S at $\boldsymbol{p}$, we consider a curve $\boldsymbol{\alpha}(t)$ such that $\boldsymbol{\alpha}(0) = \boldsymbol{p}$. Let $\boldsymbol{t} := \boldsymbol{t}(0)$ be its unit tangent vector and $\boldsymbol{k} := \boldsymbol{k}(0)$ its curvature vector at $\boldsymbol{p}$. We can write $\boldsymbol{k}$ as a sum of a normal and a tangential component, i.e.

$$\boldsymbol{k} \;=\; \boldsymbol{k}_n + \boldsymbol{k}_g \;=\; \kappa_n\,\boldsymbol{N} + \kappa_g\,\boldsymbol{v} \tag{1}$$

with a unique vector of unit length $\boldsymbol{v} \in T_{\boldsymbol{p}}(S)$. $\boldsymbol{k}_n$ is called the normal curvature vector, $\boldsymbol{k}_g$ is known as the geodesic curvature vector (see figure 2). Their signed lengths κ_n, κ_g are called normal curvature and geodesic curvature in direction of $\boldsymbol{\alpha}'(0)$ respectively. Note that the signs of κ_n and κ_g depend on the orientation of the surface S given by the normal vector $\boldsymbol{N}$.

Now the normal curvature κ_n varies with each direction $\boldsymbol{\alpha}'$. The extreme values κ_{min} and κ_{max} (it can be shown, that there exist two not necessarily distinct values) under variation of direction are called principal curvature values. Their product $K := \kappa_{min}\,\kappa_{max}$ is called Gaussian curvature and $H := \frac{1}{2}(\kappa_{min} + \kappa_{max})$ is known as mean curvature. All of these entities can be computed in terms of the fundamental forms:

$$\begin{aligned} \kappa_{min} &= H - \sqrt{H^2 - K}, \quad \kappa_{max} = H + \sqrt{H^2 - K} \\ K &= \frac{L\,N - M^2}{E\,G - F^2}, \quad H = \frac{2\,F\,M - E\,N - G\,L}{2\,(E\,G - F^2)} \end{aligned}$$

2.2 Geodesic Curves

Definition 2.1

*Let $U \subset S$ be an open set of the regular surface S. A map which assigns every point $\boldsymbol{p} \in U$ a vector $\boldsymbol{w}(\boldsymbol{p}) \in T_{\boldsymbol{p}}(S)$ is called a **vector field** in U. If for a fixed parametrization $\boldsymbol{r}(u,v)$ the coefficients a, b of the representation $\boldsymbol{w}(\boldsymbol{p}) = a\,\boldsymbol{r}_u + b\,\boldsymbol{r}_v$ are differentiable functions, $\boldsymbol{w}$ is called a **differentiable vector field**.*

If we consider a parametrized curve $\boldsymbol{\alpha}(t)$ on S together with a vector field $\boldsymbol{w}$ on S, $\boldsymbol{w}(t)$ denotes the restriction of the field to the curve $\boldsymbol{\alpha}$.

Definition 2.2

*Let $\boldsymbol{\alpha}(t)$ be a curve on S and $\boldsymbol{w}(t)$ be a differentiable vector field along $\boldsymbol{\alpha}$. The vector obtained by the normal projection of $(d\boldsymbol{w}/dt)(t_0)$ on the tangential plane $T_{\boldsymbol{\alpha}(t_0)}(S)$ is called the **covariant derivative** of $\boldsymbol{w}$ at point $\boldsymbol{\alpha}(t_0)$. This derivative is denoted by $(D\boldsymbol{w}/dt)(t_0)$.*

Now every vector field $\boldsymbol{w}(t)$ along a parametrized curve $\boldsymbol{\alpha}(t) = \boldsymbol{r}(u(t), v(t))$ can be written as

$$\boldsymbol{w}(t) = a(u(t), v(t))\,\boldsymbol{r}_u + b(u(t), v(t))\,\boldsymbol{r}_v = a(t)\,\boldsymbol{r}_u + b(t)\,\boldsymbol{r}_v.$$

Some computations yield

$$\begin{aligned} \frac{D\boldsymbol{w}}{dt} &= \left(a' + \Gamma_{11}^1\,a\,u' + \Gamma_{12}^1\,a\,v' + \Gamma_{21}^1\,b\,u' + \Gamma_{22}^1\,b\,v'\right)\boldsymbol{r}_u \\ &\quad + \left(b' + \Gamma_{11}^2\,a\,u' + \Gamma_{12}^2\,a\,v' + \Gamma_{21}^2\,b\,u' + \Gamma_{22}^2\,b\,v'\right)\boldsymbol{r}_v. \end{aligned} \tag{2}$$

The coefficients Γ_{ij}^k in (2) are differentiable functions of the parameters u and v. They are known as ***Christoffel symbols***. They may be computed as the solutions of the following system of linear equations

$$\left.\begin{aligned} &\begin{cases} \Gamma_{11}^1\,E + \Gamma_{11}^2\,F = \langle \boldsymbol{r}_{uu}, \boldsymbol{r}_u \rangle \\ \Gamma_{11}^1\,F + \Gamma_{11}^2\,G = \langle \boldsymbol{r}_{uu}, \boldsymbol{r}_v \rangle \end{cases} \\ &\begin{cases} \Gamma_{12}^1\,E + \Gamma_{12}^2\,F = \langle \boldsymbol{r}_{uv}, \boldsymbol{r}_u \rangle \\ \Gamma_{12}^1\,F + \Gamma_{12}^2\,G = \langle \boldsymbol{r}_{uv}, \boldsymbol{r}_v \rangle \end{cases} \\ &\begin{cases} \Gamma_{22}^1\,E + \Gamma_{22}^2\,F = \langle \boldsymbol{r}_{vv}, \boldsymbol{r}_u \rangle \\ \Gamma_{22}^1\,F + \Gamma_{22}^2\,G = \langle \boldsymbol{r}_{vv}, \boldsymbol{r}_v \rangle \end{cases} \end{aligned}\right\} \tag{3}$$

Note that these equations have been grouped into three pairs of independent equations each having the determinant $E\,G - F^2 \neq 0$, provided the surface parametrization $\boldsymbol{r}$ is regular. Furthermore, these equations determine all Christoffel symbols since they are symmetric with respect to their lower indices, i.e. $\Gamma_{ij}^k = \Gamma_{ji}^k$.

In our context the most frequently used vector field along $\boldsymbol{\alpha}$ will be $\boldsymbol{w}(t) = \boldsymbol{\alpha}'(t)$. We assume that $\boldsymbol{\alpha}$ is parametrized by its arc length. Hence $(d\boldsymbol{w}/dt)(t_0) = \boldsymbol{\alpha}''(t_0)$ is the curvature vector of $\boldsymbol{\alpha}$ and the covariant derivative yields its tangential component (see also equation (1)). For this reason

$$\frac{D\boldsymbol{\alpha}'}{dt}(t_0) \;=\; \kappa_g \;(\boldsymbol{N} \wedge \alpha'(t_0)) \tag{4}$$

holds which shows the connection between covariant derivatives and geodesic curvature. Intuitively $D\boldsymbol{\alpha}'/dt$ gives us informations about acceleration and curvature of $\boldsymbol{\alpha}$ as it can be seen from the surface S itself. In the planar case lines as shortest paths between two points can be characterized as curves without curvature. Thus one may wish to study curves on arbitrary surfaces which have no geodesic curvature. This leads to the following definition.

Definition 2.3

A curve $\boldsymbol{\gamma}(s)$ on the parametric surface S is called a ***geodesic curve****, if for every parameter s the geodesic curvature of $\boldsymbol{\gamma}$ equals 0.*

Using equation (4) we can give an equivalent formulation of definition 2.3. Since every regular curve can be reparametrized by its arc length, it is no restriction to assume that $\boldsymbol{\gamma}(s)$ is already parametrized by arc length, i.e. $\|\boldsymbol{\gamma}'(s)\| = 1$. Now $\boldsymbol{\gamma}(s)$ is a geodesic curve, if and only if

$$\frac{D\boldsymbol{\gamma}'}{ds}(s_0) \;=\; 0$$

for every parameter s_0. Furthermore, using equation (2) yields the essential part of the following theorem.

Theorem 2.4

For every point $\boldsymbol{p}$ of the regular parametric surface S and every vector $\boldsymbol{v} \in T_{\boldsymbol{p}}(S)$, $\boldsymbol{v} \neq 0$, there exists an $\varepsilon > 0$ and a unique geodesic curve $\boldsymbol{\gamma}(s) \subset S$, $s \in (-\varepsilon, \varepsilon)$ such that $\boldsymbol{\gamma}(0) = \boldsymbol{p}$ and $\boldsymbol{\gamma}'(0) = \boldsymbol{v}$. If we represent $\boldsymbol{\gamma}(s) = \boldsymbol{r}(u(s), v(s))$ by its curve $(u(s), v(s))$ in the parameter space, this curve satisfies the system of differential equations

$$\left.\begin{aligned} u'' + \Gamma^1_{11}\,(u')^2 + 2\,\Gamma^1_{12}\,u'v' + \Gamma^1_{22}\,(v')^2 = 0 \\ v'' + \Gamma^2_{11}\,(u')^2 + 2\,\Gamma^2_{12}\,u'v' + \Gamma^2_{22}\,(v')^2 = 0 \end{aligned}\right\} \tag{5}$$

For a proof of the theorem, see for example [dC76] p. 254 or [Sto69] p. 160.

Proposition 2.5

For every vector field $\boldsymbol{w}(t)$ along $\boldsymbol{\alpha}(t)$ with vanishing covariant derivatives, $\|\boldsymbol{w}(t)\|$ is constant.

Proof:

The assumption $D\boldsymbol{w}/dt(t_0) = 0$ means that $d\boldsymbol{w}/dt(t_0)$ is orthogonal to the tangential plane of S in $\boldsymbol{\alpha}(t_0)$. In particular we have

$$\left\langle \boldsymbol{w}(t), \frac{d\boldsymbol{w}}{dt}(t) \right\rangle \;=\; 0.$$

For this reason

$$\frac{d}{dt}\left(\|\boldsymbol{w}(t)\|^2\right) \;=\; 2\left\langle \boldsymbol{w}(t), \frac{d\boldsymbol{w}}{dt}(t) \right\rangle \;=\; 0$$

holds and $\|\boldsymbol{w}(t)\|$ is constant.

■

As an immediate consequence of proposition 2.5 we get

Corollary 2.6
Every geodesic curve $\boldsymbol{\gamma}(s)$ has constant tangent length. The parameter s is proportional to its arc length.

Corollary 2.6 is of practical interest, since it states that the solution $\boldsymbol{\gamma}(s) = \boldsymbol{r}(u(s), v(s))$ of equation (5) is already parametrized by its arc length, provided the initial values for $(u', v') = (u'(0), v'(0))$ are chosen such that

$$\|\boldsymbol{\gamma}'(0)\| \;=\; \|u'\,\boldsymbol{r}_u + v'\,\boldsymbol{r}_v\| \;=\; 1$$

holds. System (5) is transformed into a system of first order differential equations, which can be solved numerically using a standard package such as NAG (see [NAG89]):

$$\left.\begin{aligned} u_1' &= u_2 \\ u_2' &= -\left(\Gamma^1_{11}\,u_2^2 + 2\,\Gamma^1_{12}\,u_2 v_2 + \Gamma^1_{22}\,v_2^2\right) \\ v_1' &= v_2 \\ v_2' &= -\left(\Gamma^2_{11}\,u_2^2 + 2\,\Gamma^2_{12}\,u_2 v_2 + \Gamma^2_{22}\,v_2^2\right) \end{aligned}\right\} \tag{6}$$

2.3 Geodesic Offset Curves

It can be shown (see [dC76]) that any shortest path connecting two distinct points $\boldsymbol{p}$ and $\boldsymbol{q}$ on a surface has vanishing geodesic curvature. Unfortunately, a geodesic curve connecting these two points is not necessarily the shortest join between them. For example consider $\boldsymbol{p}$ and $\boldsymbol{q}$ on the sphere to be not antipodean. Since great circles are geodesics here, there are two different ways to connect $\boldsymbol{p}$ and $\boldsymbol{q}$ by geodesics. As $\boldsymbol{p}$ and $\boldsymbol{q}$ are not antipodean to each other, we have one „short“ and one „long“ geodesic connecting them. However, the following classical result holds.

Proposition 2.7
Let $\boldsymbol{p}$ be a point on the surface S. Then there exists a neighbourhood $U \subset S$ of $\boldsymbol{p}$, such that every geodesic $\boldsymbol{\gamma} : I \to U$, $\boldsymbol{\gamma}(0) = \boldsymbol{p}$, is the minimal join between $\boldsymbol{p}$ and $\boldsymbol{\gamma}(t_0) \in U$ for every $t_0 \in I$.

A proof can be found in [dC76]. Roughly spoken proposition 2.7 claims that geodesics are local shortest paths. For this reason the following definition of offset curves on surfaces makes use of geodesics.

Definition 2.8
Let $\boldsymbol{\alpha}(t)$, $t \in I \subset \mathbb{R}$ be a so-called progenitor curve on the regular surface S. For every $t_0 \in I$ we consider the point $\boldsymbol{\gamma}_{t_0}(s)$. Here $\boldsymbol{\gamma}_{t_0}$ denotes the arc length parametrized geodesic curve emanating from $\boldsymbol{\alpha}(t_0)$ in direction orthogonal to $\boldsymbol{\alpha}'(t_0)$ requiring $\boldsymbol{\alpha}'(t_0), \boldsymbol{\gamma}_{t_0}'(0), \boldsymbol{N}$ to have positive orientation. The set

$$\{\boldsymbol{\gamma}_{t_0}(s) \mid t_0 \in I\} \subset S$$

*is called **(geodesic) offset curve** of the progenitor curve $\boldsymbol{\alpha}$ at geodesic distance s.*

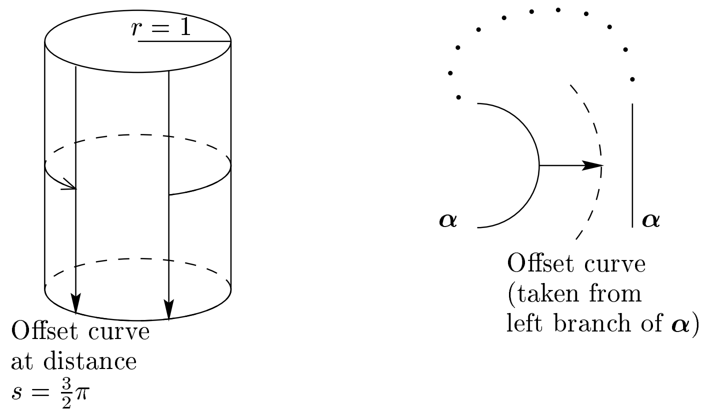


Figure 3: Global minimal join does not coincide with considered geodesic.

The geodesic distance in definition 2.8 may have a negative sign. Obviously the sign of s determines whether the offset curve lies on the left-hand side (positive sign) or on the right-hand side (negative sign) with respect to the surface orientation induced by the surface normals. Furthermore, the distance s can either be a constant or a function $s(t)$ depending on the progenitor curve's parameter t.

The notation introduced by definition 2.8 must be treated with some care. The distance between a single offset point $\boldsymbol{\gamma}_{t_0}(s)$ and the point set generated by the progenitor curve $\boldsymbol{\alpha}(t)$ may be less than s. Consider e.g. the cylinder with radius 1 along with a meridian as progenitor curve (see left half of figure 3). This „accident“ may even happen in the planar case (see right half of figure 3). In either cases the geodesic curve $\boldsymbol{\gamma}_{t_0}$ from $\boldsymbol{\alpha}(t_0)$ to the offset point is not the minimal join from $\boldsymbol{\gamma}_{t_0}(s)$ to the progenitor curve.

Proposition 2.9

The geodesic offset curve of a given progenitor curve $\boldsymbol{\alpha}(t)$ at distance s_0 can be represented as a parametrized curve $\boldsymbol{\alpha}_{s_0}(t)$.

Proof:

Every offset point $\boldsymbol{\gamma}_{t_0}(s_0)$ is the solution of system (6), which itself has a differentiable right side. By definition 2.8 the offset curve may be obtained by the differentiable variation of t (namely along $\boldsymbol{\alpha}(t)$) of the initial values of system (6); note that those initial conditions change differentiably with progenitor curve's normals. By a classical result of the theory of ordinary differential equations (see [Wal93]) the solutions obtained by this variation is a differentiable function of t.

■

Note that the offset curve is not necessarily regular. For instance, consider the curve (t, t^2) and its offset curve at distance 1 in the planar case. In this example the offset curve has a singular point for $t = 0$. This motivates the subsequent definition.

Definition 2.10

A surface point $\boldsymbol{p} \in S$ is called a ***focal point*** *of the given progenitor curve $\boldsymbol{\alpha}$, if it is a singular point of a geodesic offset curve $\boldsymbol{\alpha}_s(t)$ at a certain distance s, i.e. $\boldsymbol{\alpha}'_s(t) = 0$.*

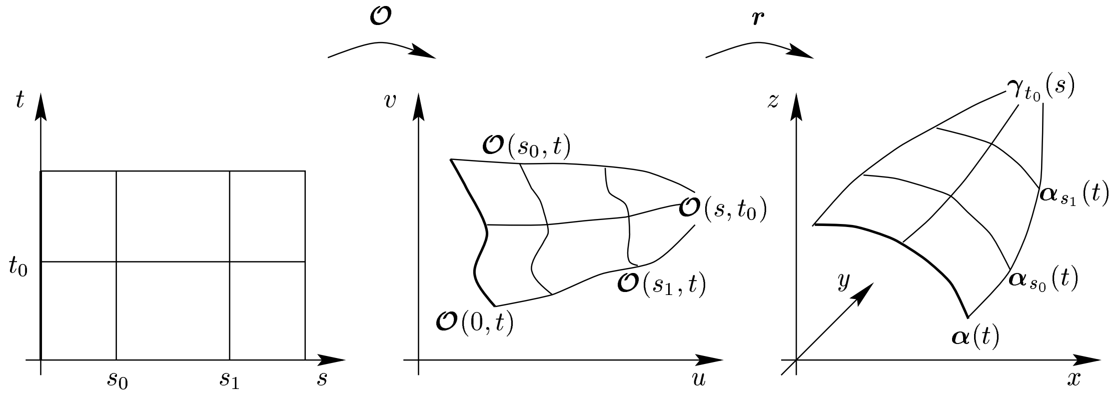


Figure 4: Offset function $\mathcal{O}(s,t)$ of the given progenitor curve $\boldsymbol{\alpha}(t)$.

3 Offset Functions and Jacobi Fields

Definition 3.1
Let S be a regular parametric surface parametrized by $\boldsymbol{r}(u,v)$ and let $\boldsymbol{\alpha}(t)$ be a progenitor curve. The function $\mathcal{O} : (s,t) \mapsto (u,v)$ defined by

$$\boldsymbol{r}\left(\mathcal{O}(s,t)\right) \;=\; \boldsymbol{\gamma}_t(s) \;=\; \boldsymbol{\alpha}_s(t)$$

is called the ***(geodesic) offset function*** *on S with respect to the progenitor curve $\boldsymbol{\alpha}(t)$.*

By definition 3.1 $\mathcal{O}(s,t)$ equals the (u,v)-parameters of the offset points from the progenitor point $\boldsymbol{\alpha}(t)$ at geodesic distance s. Using this function we have a uniform tool to describe all the curves defined above (see also figure 4):

$$\begin{aligned}
\boldsymbol{r}\left(\mathcal{O}(0,t)\right) &= \boldsymbol{\alpha}(t) && \text{(progenitor curve)}\\
\boldsymbol{r}\left(\mathcal{O}(s,t_0)\right) &= \boldsymbol{\gamma}_{t_0}(s) && \text{(geodesic curve starting at } \boldsymbol{\alpha}(t_0))\\
\boldsymbol{r}\left(\mathcal{O}(s_0,t)\right) &= \boldsymbol{\alpha}_{s_0}(t) && \text{(offset curve at geodesic distance } s_0).
\end{aligned}$$

Proposition 3.2
The geodesic offset function is differentiable. Its partial derivatives $\partial_s\mathcal{O}$ and $\partial_t\mathcal{O}$ are given by

$$\boldsymbol{\gamma}'_{t_0}(s) \;=\; (\boldsymbol{r}_u, \boldsymbol{r}_v)\cdot \partial_s\mathcal{O} \tag{7}$$

$$\boldsymbol{\alpha}'_{s_0}(t) \;=\; (\boldsymbol{r}_u, \boldsymbol{r}_v)\cdot \partial_t\mathcal{O}, \tag{8}$$

where $(\boldsymbol{r}_u, \boldsymbol{r}_v)$ is the Jacobian matrix of $\boldsymbol{r}$.

Proof:
The function $\mathcal{O}(s,t)$ describes the solutions of the geodesic differential equation (5) with its initial values depending differentiably on the parameters s, t. Thus $\mathcal{O}(s,t)$ is a differentiable function. Equations (7) and (8) follow immediately by applying the chain rule.

■

Note: by solving the system of differential equations (5) not only the parameters $(u(s), v(s))$ of the geodesic curve $\boldsymbol{\gamma}_{t_0}(s) = \boldsymbol{r}(u(s), v(s))$ are computed. The derivative $\boldsymbol{\gamma}'_{t_0}(s) = u'\,\boldsymbol{r}_u + v'\,\boldsymbol{r}_v$ can be obtained too. Clearly the partial derivative $\partial_s \boldsymbol{\mathcal{O}} = (u', v')$ is already computed by solving the geodesic equation. Since $\boldsymbol{\alpha}'_{s_0}(t)$ can not be computed directly, we are not able to exploit equation (8) yet. In order to compute the partial derivative $\partial_t \boldsymbol{\mathcal{O}}$, we introduce the vector field

$$\boldsymbol{J}_{t_0}(s) \;:=\; \boldsymbol{\alpha}'_s(t_0) \tag{9}$$

of the offset curve's tangent vectors along any geodesic $\boldsymbol{\gamma}_{t_0}(s)$.

Often it will turn out to be very useful to regard the given surface as parametrized locally by the parameters s and t. Given a progenitor curve $\boldsymbol{\alpha}(t)$ this parametrization is introduced by

$$\widetilde{\boldsymbol{r}}(s,t) \;:=\; (\boldsymbol{r} \circ \boldsymbol{\mathcal{O}})\,(s,t). \tag{10}$$

The isoparametric lines here are the geodesic curves $\boldsymbol{\gamma}_t(s)$ and the offset curves $\boldsymbol{\alpha}_s(t)$ respectively. The Jacobian matrix here is given by $(\boldsymbol{r}_u, \boldsymbol{r}_v)(\partial_s \boldsymbol{\mathcal{O}}, \partial_t \boldsymbol{\mathcal{O}})$, which is one-to-one, provided one stays away from focal points of the progenitor curve. Hence the parametrization $\widetilde{\boldsymbol{r}}$ is regular in a some neighbourhood of $\boldsymbol{\alpha}$.

Definition 3.3

Let $\boldsymbol{\alpha}(s)$, $s \in [0, l]$ be a curve parametrized by its arc length on the surface S. A differentiable map $h : [0, l] \times (-\varepsilon, \varepsilon) \to S$ such that $\boldsymbol{h}(s, 0) \;=\; \boldsymbol{\alpha}(s)$ is called a **variation** *of $\boldsymbol{\alpha}$.*

Definition 3.4

Let $\boldsymbol{\gamma}(s)$ be a geodesic curve parametrized by its arc length on S and let $h : [0, l] \times (-\varepsilon, \varepsilon) \to S$ be a variation of $\boldsymbol{\gamma}$ with the additional property that for every $t_0 \in (-\varepsilon, \varepsilon)$ the curve $\boldsymbol{h}_{t_0}(s) = \boldsymbol{h}(s, t_0)$ is a parametrized geodesic. (Note that the curve $\boldsymbol{h}_{t_0}(s)$ is not necessarily parametrized by its arc length.) The vector field $(\partial \boldsymbol{h}/\partial t)(s, 0)$ is called a **Jacobi field** *along $\boldsymbol{\gamma}$.*

Obviously the vector field introduced in (9) is a Jacobi field along $\boldsymbol{\gamma}_{t_0}$. Next we give a classical result on Jacobi fields which can be used to characterize them by an analytical condition.

Proposition 3.5

A vector field $\boldsymbol{w}(s)$ along a geodesic $\boldsymbol{\gamma}$ is a Jacobi field if and only if it satisfies

$$\frac{D^2}{ds^2}\boldsymbol{w}(s) + K(s)\;(\boldsymbol{\gamma}'(s) \wedge \boldsymbol{w}(s)) \wedge \boldsymbol{\gamma}'(s) \;=\; 0 \tag{11}$$

where $K(s)$ is the Gaussian curvature of the surface S at $\boldsymbol{\gamma}(s)$.

A proof of proposition 3.5 can be found in [dC76]. Equation (11) will be the key to the method considered in this paper. Before we explain this in more detail, we state a quite useful property of the special Jacobi field $\boldsymbol{J}_{t_0}(s)$ considered here, namely that all vectors are orthogonal to the geodesics tangent vectors $\boldsymbol{\gamma}'_{t_0}(s)$.

Proposition 3.6

The Jacobi field $\boldsymbol{J}_{t_0}(s)$ of the offset curve's tangent vectors as introduced in equation (9) is orthogonal to the tangent vectors of the geodesic $\boldsymbol{\gamma}_{t_0}$, i.e.

$$\langle \boldsymbol{J}_{t_0}(s), \boldsymbol{\gamma}'_{t_0}(s) \rangle \;=\; 0$$

holds for all s.

Proof:
Consider the parametrization $\widetilde{\boldsymbol{r}}$ as introduced in (10) in a neighbourhood of $\boldsymbol{\alpha}$. The coefficients $\widetilde{E}$ and $\widetilde{G}$ of the first fundamental form with respect to the local coordinate system $\widetilde{\boldsymbol{r}}_s$, $\widetilde{\boldsymbol{r}}_t$ are given by

$$\widetilde{E} \;=\; \langle \boldsymbol{\gamma}'_{t_0}(s), \boldsymbol{\gamma}'_{t_0}(s)\rangle \;=\; 1, \quad \widetilde{G} \;=\; \langle \boldsymbol{\alpha}'_{s_0}(t), \boldsymbol{\alpha}'_{s_0}(t)\rangle \;=\; \|\boldsymbol{\alpha}'_{s_0}(t)\|^2 .$$

Since the geodesics $\boldsymbol{\gamma}_{t_0}(s)$ emanating orthogonally from the given progenitor curve are isoparametric lines here, the second equation of the system (5) yields $\Gamma^2_{11}(s')^2 = 0$ implying $\Gamma^2_{11} = 0$. On the other hand

$$\left.\begin{array}{l} \Gamma^1_{11}\,\widetilde{E} + \Gamma^2_{11}\,\widetilde{F} = \langle \widetilde{\boldsymbol{r}}_{uu}, \widetilde{\boldsymbol{r}}_u\rangle = \frac{1}{2}\widetilde{E}_s \\ \Gamma^1_{11}\,\widetilde{F} + \Gamma^2_{11}\,\widetilde{G} = \langle \widetilde{\boldsymbol{r}}_{uu}, \widetilde{\boldsymbol{r}}_v\rangle = \widetilde{F}_s - \frac{1}{2}\widetilde{E}_t \end{array}\right\}$$

holds (see equation (3)). Since $\widetilde{E} = 1$ and $\Gamma^2_{11} = 0$ we have $\Gamma^1_{11} = 0 = \widetilde{F}_s$. Thus

$$\widetilde{F}(s_0, t_0) \;=\; \widetilde{F}(0, t_0) \;=\; \langle \boldsymbol{\alpha}'(t_0), \boldsymbol{\gamma}'_{t_0}(0)\rangle \;=\; 0$$

holds, as the coefficient $\widetilde{F}$ does not depend on the geodesic distance s.

■

Exploiting proposition 3.6 together with a property of the cross product, we have

$$(\boldsymbol{\gamma}'_{t_0}(s) \wedge \boldsymbol{J}_{t_0}(s)) \wedge \boldsymbol{\gamma}'_{t_0}(s) \;=\; \|\boldsymbol{\gamma}'_{t_0}(s)\|^2\, \boldsymbol{J}_{t_0}(s) - \langle \boldsymbol{J}_{t_0}(s), \boldsymbol{\gamma}'_{t_0}(s)\rangle\, \boldsymbol{\gamma}'_{t_0}(s) \;=\; \boldsymbol{J}_{t_0}(s)$$

as the geodesic $\boldsymbol{\gamma}_{t_0}(s)$ is proposed to be parametrized by its arc length. Hence, in the special case of the Jacobi field $\boldsymbol{J}_{t_0}(s)$, equation (11) yields the identity

$$\frac{D^2}{ds^2}\boldsymbol{J}_{t_0}(s) + K(s)\,\boldsymbol{J}_{t_0}(s) \;=\; 0. \tag{12}$$

If we are only interested in the behaviour of the length $y_{t_0}(s) := \|\boldsymbol{J}_{t_0}(s)\|$ of this Jacobi field, a straightforward simplification of equation (12) yields

$$y''_{t_0}(s) + K(s)\,y_{t_0}(s) \;=\; 0, \tag{13}$$

which is an ordinary differential equation of second order.

Proposition 3.7

Let $\boldsymbol{r}(u,v)$ be an orthogonal parametrization of a neighbourhood of the regular oriented surface S, i.e. $F = \langle \boldsymbol{r}_u, \boldsymbol{r}_v\rangle = 0$. Furthermore, let $\boldsymbol{\alpha}(t) = \boldsymbol{r}(u(t), v(t))$ be a curve on S and $\boldsymbol{w}(t)$ a vector field of unit length along $\boldsymbol{\alpha}(t)$. If $\varphi(t)$ denotes the angle formed by $\boldsymbol{r}_u(u(t),v(t))$ and $\boldsymbol{w}(t)$ with respect to the given orientation,

$$\left\langle \frac{d\boldsymbol{w}}{dt}(t), (\boldsymbol{N} \wedge \boldsymbol{w}(t))\right\rangle \;=\; \frac{1}{2\sqrt{E\,G}}\;\{G_u\,v' - E_v\,u'\} + \frac{d}{dt}\varphi(t)$$

holds.

A proof of proposition 3.7 can be found in [dC76]. Since $\boldsymbol{w}(t) \in T_{\boldsymbol{p}}(S)$ is assumed to be of unit length, $\boldsymbol{w}(t)$ is orthogonal to either $(d\boldsymbol{w}/dt)(t)$ and to its normal projection onto the tangent plane. Thus $(D\boldsymbol{w}/dt)(t)$ must be a vector in direction of $(\boldsymbol{N} \wedge \boldsymbol{w}(t))$. Furthermore, the value of $\langle (d\boldsymbol{w}/dt)(t), (\boldsymbol{N} \wedge \boldsymbol{w}(t))\rangle$ yields its signed length with respect to the orientation introduced by the surface normal.

Proposition 3.8
For every geodesic $\gamma_{t_0}(s)$ the tangent length $y_{t_0}(s) = \|\alpha'_s(t_0)\|$ of geodesic offset curves along $\gamma_{t_0}(s)$ satisfies

$$y'_{t_0}(s) \;=\; -\kappa_g\, y_{t_0}(s),$$

provided $\gamma_{t_0}(s)$ is not a focal point of the given progenitor curve $\alpha(t)$. Here, κ_g denotes the geodesic curvature of α_s at point $\alpha_s(t_0) = \gamma_{t_0}(s)$.

Proof:
In a sufficiently small neighbourhood of the non-focal point $\gamma_{t_0}(s_0)$ the surface can be considered as parametrized by $\widetilde{\boldsymbol{r}} = \boldsymbol{r} \circ \mathcal{O}$. By proposition 3.6 this is a regular, orthogonal parametrization. The coefficients of the first fundamental form are given by

$$\widetilde{E} \;=\; 1, \quad \widetilde{F} \;=\; 0, \quad \widetilde{G} \;=\; \langle \alpha'_s(t), \alpha'_s(t) \rangle \;=\; (y_t(s))^2\,,$$

since geodesics can be chosen to be parametrized by their arc length (see corollary 2.6). Along any offset curve $\alpha_s(t)$ we study the vector field

$$\boldsymbol{w}(t) \;:=\; \frac{\alpha'_s(t)}{\|\alpha'_s(t)\|}$$

which obviously is of unit length. Hence applying proposition 3.7 yields

$$\left\langle \frac{d\boldsymbol{w}}{dt}(t), (\boldsymbol{N} \wedge \boldsymbol{w}(t)) \right\rangle \;=\; -y'_t(s). \tag{14}$$

with respect to the surface orientation. On the other hand the derivation of the vector field is given by

$$\frac{d\boldsymbol{w}}{dt}(t) \;=\; \frac{1}{\|\alpha'_s(t)\|}\,\alpha''_s(t) - \frac{\langle \alpha'_s(t), \alpha''_s(t) \rangle}{\|\alpha'_s(t)\|^3}\,\alpha'_s(t).$$

If we compare this vector with the curvature vector $\boldsymbol{k}$ of α_s, which is given by

$$\boldsymbol{k}(t) \;=\; \frac{1}{\|\alpha'_s(t)\|^2}\,\alpha''_s(t) - \frac{\langle \alpha'_s(t), \alpha''_s(t) \rangle}{\|\alpha'_s(t)\|^4}\,\alpha'_s(t),$$

we observe

$$\frac{d\boldsymbol{w}}{dt}(t) \;=\; y_t(s)\,\boldsymbol{k}(t),$$

i.e. the derivative of the vector field $(d\boldsymbol{w}/dt)$ equals the curvature vector of α_s up to the factor $y'_s(t)$. Since the covariant derivative of $\boldsymbol{w}(t)$ can be obtained by normal projection of $(d\boldsymbol{w}/dt)$ onto the surface's tangent plane, we observe

$$\left\langle \frac{d\boldsymbol{w}}{dt}(t), (\boldsymbol{N} \wedge \boldsymbol{w}(t)) \right\rangle \;=\; y_t(s)\,\kappa_g,$$

which (together with equation (14)) proofs the claimed identity.

■

Another proof of proposition 3.8 can be found in [BL73], p. 205. Now we are able to solve the differential equation $y''_{t_0}(s) = -K(s)\,y_{t_0}(s)$ proposed in equation (13). Once the equation is transformed into the corresponding system of first order equations

$$\left.\begin{aligned} y'_1(s) &= y_2(s) \\ y'_2(s) &= -K(s)\,y_1(s) \end{aligned}\right\} \tag{15}$$

standard packages such as NAG (see [NAG89]) can be used to compute $y_1(s) = y_{t_0}(s)$ and $y_2(s) = y'_{t_0}(s)$ numerically. Appropriate initial values can be achieved by

$$y_1(0) \;=\; \|\boldsymbol{\alpha}'(t_0)\|, \quad y_2(0) \;=\; -\kappa_g\, y_1(0)$$

exploiting proposition 3.8. Moreover, with the knowledge of the signed length $y_{t_0}(s)$ of the geodesic offset curve $\boldsymbol{\alpha}_s$ at parameter t_0, the tangent vector $\boldsymbol{\alpha}'_{t_0}(s)$ is uniquely determined (see proposition 3.6). Thus, both partial derivatives $\partial_s\boldsymbol{\mathcal{O}}$ and $\partial_t\boldsymbol{\mathcal{O}}$ of the offset function can be achieved by equation (7) and (8) respectively.

4 Medial Curves

As a first application of the methods developed so far, we present a method to compute the medial curve of two border curves on the surface S. Let $\boldsymbol{\alpha}(t), \widetilde{\boldsymbol{\alpha}}(\widetilde{t})$ be two regular curves on the parametrized surface S. The offset functions belonging to these curves shall be denoted by $\boldsymbol{\mathcal{O}}(s,t)$ and $\widetilde{\boldsymbol{\mathcal{O}}}(\widetilde{s},\widetilde{t})$ respectively. Consider the vector-valued function $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$\boldsymbol{F}(s,t,\widetilde{t}) \;:=\; \widetilde{\boldsymbol{\mathcal{O}}}(\widetilde{\sigma}s,\widetilde{t}) - \boldsymbol{\mathcal{O}}(\sigma s,t)$$

where $\sigma, \widetilde{\sigma} \in \{-1,1\}$ are constant signs, which control to which side each offset shall be computed, whilst $s \geq 0$ by convention. Furthermore, let $(s_0,t_0,\widetilde{t}_0)$ be a triple, for which

$$\boldsymbol{F}(s_0,t_0,\widetilde{t}_0) \;=\; \boldsymbol{0}, \quad \det\Big(\partial_s\boldsymbol{F}(s_0,t_0,\widetilde{t}_0), \partial_{\widetilde{t}}\boldsymbol{F}(s_0,t_0,\widetilde{t}_0)\Big) \;\neq\; 0 \tag{16}$$

holds. The implicit function theorem implies that there exists a neighbourhood $J \subset I$ of t_0 and differentiable functions $s, \varphi : J \to \mathbb{R}$ such that

$$\boldsymbol{F}(s(t),t,\varphi(t)) \;=\; \boldsymbol{0}$$

holds for every $t \in J$. Let us denote the implicitly defined curve by $\boldsymbol{m}(t)$. This is a curve in the parameter space of the surface S with the property

$$\boldsymbol{\mathcal{O}}(\sigma s(t),t) \;=\; \boldsymbol{m}(t) \;=\; \widetilde{\boldsymbol{\mathcal{O}}}(\widetilde{\sigma}s(t),\varphi(t)) \tag{17}$$

for every $t \in J$. The corresponding curve $\boldsymbol{r}(\boldsymbol{m}(t)) =: \boldsymbol{\mu}(t)$ on S fulfills

$$\boldsymbol{\gamma}_t(\sigma s(t)) \;=\; \boldsymbol{\mu}(t) \;=\; \widetilde{\boldsymbol{\gamma}}_{\varphi(t)}(\widetilde{\sigma}s(t)), \quad t \in J.$$

In other words: $\boldsymbol{\mu}(t)$ lies at geodesic distance $s(t)$ to the border points $\boldsymbol{\alpha}(t)$ and $\widetilde{\boldsymbol{\alpha}}(\varphi(t))$, where the distance is taken along the geodesic $\boldsymbol{\gamma}_t$ and $\widetilde{\boldsymbol{\gamma}}_{\varphi(t)}$ respectively (see figure 5). Note that these geodesics are not necessarily global minimal joins to the given border curves, but only local minimal paths.

The condition of regularity in equation (16) in more detail is $\det\Big(\widetilde{\sigma}\,\partial_s\widetilde{\boldsymbol{\mathcal{O}}} - \sigma\,\partial_s\boldsymbol{\mathcal{O}}, \partial_t\widetilde{\boldsymbol{\mathcal{O}}}\Big) \neq 0$. Without loss of generality we can assume that this condition holds for all $t \in J$ (maybe after reducing J). Differentiating equation (17) yields

$$\Big(\widetilde{\sigma}\,\partial_s\widetilde{\boldsymbol{\mathcal{O}}} - \sigma\,\partial_s\boldsymbol{\mathcal{O}}, \partial_t\widetilde{\boldsymbol{\mathcal{O}}}\Big)\begin{pmatrix} s' \\ \varphi' \end{pmatrix} \;=\; \partial_t\boldsymbol{\mathcal{O}}. \tag{18}$$

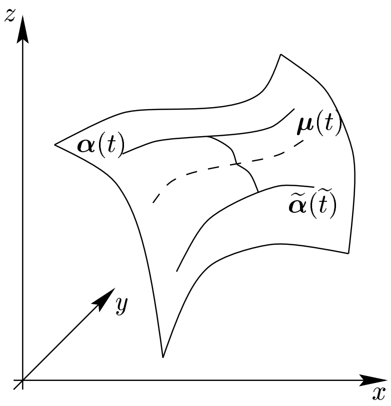


Figure 5: Medial curve $\boldsymbol{\mu}(t)$ of two border curves $\boldsymbol{\alpha}(t)$, $\widetilde{\boldsymbol{\alpha}}(\widetilde{t})$.

Since the matrix is assumed to be regular for all $t \in J$, we have

$$\begin{pmatrix} s' \\ \varphi' \end{pmatrix} = \left(\widetilde{\sigma}\,\partial_s\widetilde{\boldsymbol{\mathcal{O}}} - \sigma\,\partial_s\boldsymbol{\mathcal{O}}, \partial_t\widetilde{\boldsymbol{\mathcal{O}}}\right)^{-1} \partial_t\boldsymbol{\mathcal{O}}. \tag{19}$$

This is a system of ordinary differential equations which can be used to trace the medial curve introduced above, provided initial values can be found. Note that the partial derivatives of the geodesic offset functions $\boldsymbol{\mathcal{O}}$, $\widetilde{\boldsymbol{\mathcal{O}}}$ can be computed using the techniques developed in the preceding section.

System (18) becomes singular, if and only if the matrix is singular or if the vector on the right-hand side equals zero. The latter occurs if and only if the medial curve is marching through a focal point of the border curve $\boldsymbol{\alpha}$. For the remaining case assume there exists a real number λ such that

$$\widetilde{\sigma}\,\partial_s\widetilde{\boldsymbol{\mathcal{O}}} - \sigma\,\partial_s\boldsymbol{\mathcal{O}} = \lambda\,\partial_t\widetilde{\boldsymbol{\mathcal{O}}}$$

holds. By multiplying this identity with the Jacobian matrix $M = (\boldsymbol{r}_u, \boldsymbol{r}_v)$ of the parametrization we get

$$\widetilde{\sigma}\,\widetilde{\boldsymbol{\gamma}}'_{\varphi(t)}(\widetilde{\sigma}\,s(t)) - \sigma\,\boldsymbol{\gamma}'_t(\sigma\,s(t)) = \lambda\,\widetilde{\boldsymbol{\alpha}}'_{\widetilde{\sigma}\,s(t)}(\varphi(t)). \tag{20}$$

Since $\widetilde{\boldsymbol{\gamma}}'_{\varphi(t)}$ is perdendicular to $\widetilde{\boldsymbol{\alpha}}'_s$, this implies

$$\left\langle \boldsymbol{\gamma}'_t(\widetilde{\sigma}\,s(t)), \widetilde{\boldsymbol{\gamma}}'_{\varphi(t)}(\widetilde{\sigma}\,s(t)) \right\rangle = \sigma\,\widetilde{\sigma} = \pm 1.$$

Thus the angle between the tangent vectors of the two geodesic curves $\boldsymbol{\gamma}'_t(\widetilde{\sigma}\,s(t))$ and $\widetilde{\boldsymbol{\gamma}}'_{\varphi(t)}(\widetilde{\sigma}\,s(t))$ at the medial point must be either 0 or π. Assume the angle equals 0. Then we have found two distinct geodesic curves sharing their tangent directions at $\boldsymbol{\mu}(t)$. This contradicts theorem 2.4 stating that geodesics curves are uniquely determined by their tangent direction. Consequently, the angle equals π and the signs σ and $\widetilde{\sigma}$ must have been chosen such that $\sigma\,\widetilde{\sigma} = -1$. Hence the left-hand side of equation (20) equals zero. This implies either $\lambda = 0$ (and system (18) is regular here) or $\widetilde{\boldsymbol{\alpha}}'_{\widetilde{\sigma}s(t)} = 0$. In the latter case the medial curve is marching through a focal point of the $\widetilde{\boldsymbol{\alpha}}$. Altogether we have established the following result.

Lemma 4.1
The system of differential equations (18) is singular if and only if $\partial_t \boldsymbol{\mathcal{O}}(s(t),t) = 0$ or $\partial_t \widetilde{\boldsymbol{\mathcal{O}}}(s(t), \varphi(t)) = 0$ holds, i.e. at focal points of $\boldsymbol{\alpha}(t)$ or $\widetilde{\boldsymbol{\alpha}}(\widetilde{t})$ respectively.

In case the two border curves $\boldsymbol{\alpha}(t)$ and $\widetilde{\boldsymbol{\alpha}}(\widetilde{t})$ intersect, one of these shared surface points can be used to provide initial values of differential equation (19). An example for this case can be found in figure 7. If no intersection is present, finding appropriate initial values can be regarded as the problem of finding a root of

$$\boldsymbol{G}(s,t) \;:=\; \widetilde{\boldsymbol{\mathcal{O}}}\left(\widetilde{\sigma}\, s, \widetilde{t}_0\right) - \boldsymbol{\mathcal{O}}\left(\sigma\, s, t\right) \;=\; 0. \tag{21}$$

Here the parameter $\widetilde{t}_0$ of the second border curve is considered to be constant. Since the partial derivatives of $\boldsymbol{G}$ are given by

$$\frac{\partial}{\partial s}\boldsymbol{G} \;=\; \widetilde{\sigma}\, \partial_s \widetilde{\boldsymbol{\mathcal{O}}}\left(\widetilde{\sigma}\, s, \widetilde{t}_0\right) - \sigma\, \partial_s \boldsymbol{\mathcal{O}}\left(\sigma\, s, t\right) \qquad \text{and} \qquad \frac{\partial}{\partial t}\boldsymbol{G} \;=\; -\partial_t \boldsymbol{\mathcal{O}}\left(\sigma\, s, t\right)$$

the Jacobian matrix of $\boldsymbol{G}$ is regular, provided one stays away from focal points of the first border curve $\boldsymbol{\alpha}(t)$. Hence a Newton method

$$\begin{pmatrix} s_{i+1} \\ t_{i+1} \end{pmatrix} := \begin{pmatrix} s_i \\ t_i \end{pmatrix} - \left(\frac{\partial}{\partial s}\boldsymbol{G}(s_i,t_i), \frac{\partial}{\partial t}\boldsymbol{G}(s_i,t_i)\right)^{-1} \boldsymbol{G}(s_i,t_i)$$

can be used to approximate a root of $\boldsymbol{G}$. Of course, any modified Newton method (typically included in standard packages such as NAG (see [NAG89])) can be used instead. However, most of these methods require the computation of the partial derivatives of $\boldsymbol{G}$ which are accessible with the methods presented here.

One may get some surprising perhaps unexpected results by this strategy. Consider the example in the Euclidean plane given by figure 6. Here parameter t_2 along with the corresponding distance s_2 provides a root of equation (21) as well as (s_1, t_1) does. By drawing the corresponding circles with center $\boldsymbol{\mu}(t_1)$ and $\boldsymbol{\mu}(t_2)$ and radius s_1 and s_2 respectively it is quite obvious, that $\boldsymbol{\mu}(t_2)$ does not belong to the equidistantial curve of the (global) arcs of $\boldsymbol{\alpha}$ and $\widetilde{\boldsymbol{\alpha}}$ whereas $\boldsymbol{\mu}(t_1)$ does. The

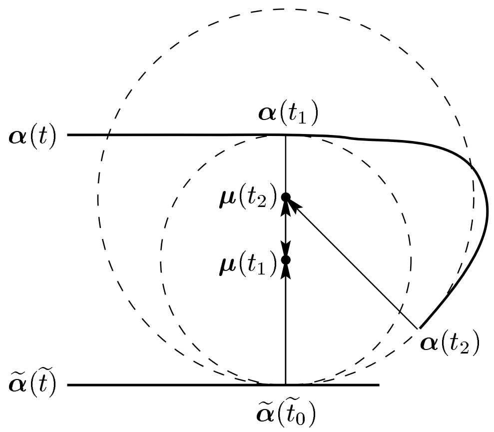


Figure 6: Problems in finding initial medial points.

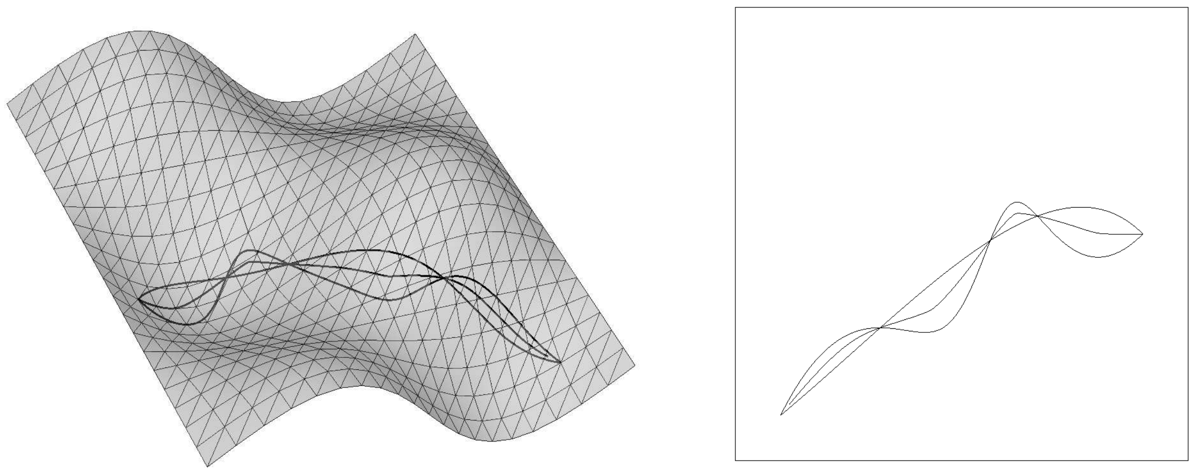


Figure 7: Medial curve on a wave-like surface and the related curves in the (u, v) parameter space.

reason for this lies in the *global* shape of curve $\boldsymbol{\alpha}(t)$. *However,* $\boldsymbol{\mu}(t_2)$ *can also be considered as a point on the equidistantial curve defined with respect to two subarcs of* $\boldsymbol{\alpha}$ *and* $\widetilde{\boldsymbol{\alpha}}$*, each contained in a small neighbourhood of* $\boldsymbol{\alpha}(t_2)$ *and* $\widetilde{\boldsymbol{\alpha}}(\widetilde{t}_0)$ *respectively.* Therefore, even in this case the algorithm yields locally a geometrically correct result. Clearly it depends on the choice of starting values for the Newton method which of the roots (if there are more than one) is approximated.

Figures 7 and 8 visualize two results obtained with our algorithm. Our implementation uses a Runge-Kutta-Merson method for integrating the medial differential equation (19) and Adams method for solving the Jacobi system (15), both routines taken from the NAG package (see [NAG89]). The considered surface in both cases is given by the parametrization $(u, v, \sin(u)\cos(v))$ over the parameter space $[0, 2\pi] \times [0, 2\pi]$. The first example (figure 7) deals with progenitor curves that intersect each other five times. Therefore one of the outer intersection points was used as a starting point for tracing the medial curve in the parameter space. The remaining intersection points were passed during the process with an accuracy of about 10^{-10}. In the right half of figure 7 the corresponding representation of progenitor curves and computed medial curve in the (u, v) parameter space are shown.

In the second example two different border curves on the same surface were considered, now without any intersections. In order to get a first estimation of the numerical behaviour, for every approximated pair of geodesic distance $s(t_0)$ at which the medial point lies and corresponding parameter $\varphi(t_0)$, the following test was carried out. We numerically solved the system of geodesic differential equations (6) for both border curves $\boldsymbol{\alpha}$ and $\widetilde{\boldsymbol{\alpha}}$ at parameters t_0 and $\widetilde{t}_0$ respectively and geodesic distance $s(t_0)$ each. This yields an approximation of the parameter values (u_0, v_0) and $(\widetilde{u}_0, \widetilde{v}_0)$ of $\boldsymbol{\gamma}_{t_0}(s(t_0))$ and $\widetilde{\boldsymbol{\gamma}}_{\varphi(t_0)}(s(t_0))$ respectively. Ideally these points would coincide since they are supposed to be medial points. Therefore the difference vector $(u_0 - \widetilde{u}_0, v_0 - \widetilde{v}_0)$ was considered to check the accuracy of our results. In the example shown in figure 8 no errors

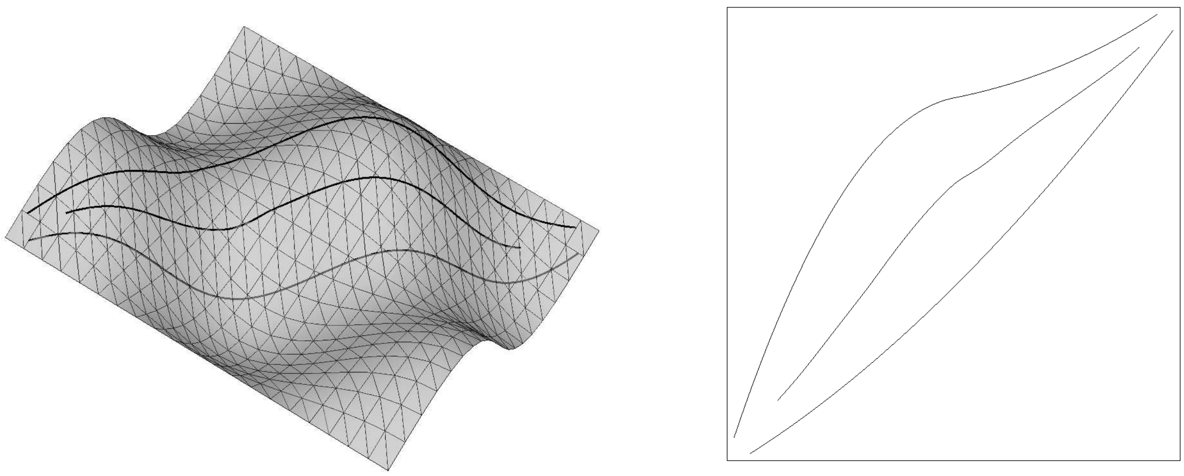


Figure 8: Medial curve on a wave-like surface and the related curves in the (u, v) parameter space.

worse than 10^{-10} occured using this test (all computations have been done in double precision arithmetic).

5 Focal Curves

As mentioned above the set of focal points plays an important role when geodesic offset curves and medial curves are considered, since they become singular if and only if they pass a focal point of a progenitor curve. Hence one of our current interests is to study the set of focal points in more detail. Below we will give a brief summary of the results we were able to derive so far.

The crucial part in tracing medial curves is the computation of $\partial_t \boldsymbol{\mathcal{O}}$ of the offset function (see section 4). We manage to achieve this partial derivative by solving the Jacobi differential equation (13) which is transformed into a system of first order differential equations (see (15))

$$\boldsymbol{Y}_t'(s) := \begin{pmatrix} y_1'(s) \\ y_2'(s) \end{pmatrix} = \begin{pmatrix} y_2(s) \\ -K(s,t)\, y_1(s) \end{pmatrix}. \tag{22}$$

The independent variable of this system is the geodesic distance s. However it has an additional parameter t. For this reason we will denote the left side of this system by $\boldsymbol{Y}(s,t) := \boldsymbol{Y}_t(s)$. The solution of this system yields

$$\boldsymbol{Y}(s,t) = \begin{pmatrix} y_t(s) \\ y_t'(s) \end{pmatrix} = \begin{pmatrix} y(s,t) \\ \frac{\partial}{\partial s} y(s,t) \end{pmatrix}.$$

In the sequel it will be important to be able to compute the partial derivative $\frac{\partial}{\partial t} y(s,t)$ as well. Exploiting the differentiable dependence on both parameters s, t, the theory of ordinary differential equations justifies that we can use the linearized system of Jacobi's equations (22) and linearized initial conditions in order to obtain the wanted partial derivative of $y(s,t)$ (see [Wal93]). For those computations we introduce the system of ordinary differential equations

$$\left.\begin{aligned} &\frac{d}{ds} z_1(s_0,t_0) = 2\, z_2(s_0,t_0) \\ &\frac{d}{ds} z_2(s_0,t_0) = -2\, K(s_0,t_0)\, z_1(s_0,t_0) - \frac{\partial}{\partial t} K(s_0,t_0)\, y(s_0,t_0) \end{aligned}\right\} \tag{23}$$

A solution $\boldsymbol{Z}(s,t) = (z_1(s,t), z_2(s,t))$ of equation (23) has the property

$$\boldsymbol{Z}(s_0,t_0) \;=\; \begin{pmatrix} \frac{\partial}{\partial t} y(s_0,t_0) \\ \frac{\partial^2}{\partial s\, \partial t} y(s_0,t_0) \end{pmatrix}. \tag{24}$$

Appropriate initial values for this system can be found using proposition 3.8

$$\left.\begin{aligned} z_1(0,t) \;&=\; \frac{\partial}{\partial t} y(0,t) \;&&=\; \frac{\langle \boldsymbol{\alpha}'(t), \boldsymbol{\alpha}''(t)\rangle}{\|\boldsymbol{\alpha}'(t)\|} \\ z_2(0,t) \;&=\; \frac{\partial^2}{\partial s \partial t} y(0,t) \;&&=\; -\left\{ \frac{\partial}{\partial t}\kappa_g(0,t)\, y(0,t) + \kappa_g(0,t)\, \frac{\partial}{\partial t} y(0,t) \right\} \end{aligned}\right\} \tag{25}$$

where $\boldsymbol{\alpha}(t)$ is the underlying progenitor curve. Note that the partial derivatives of the Gaussian curvature $\frac{\partial}{\partial t}K$ in equation (23) and of the geodesic curvature $\frac{\partial}{\partial t}\kappa_g$ in (25) are needed here. Since both formulas tend to be very large and their derivation is rather technical they shall be omitted here.

Let us now consider a focal point at parameter t_0 of the given progenitor curve $\boldsymbol{\alpha}(t)$, $t \in I$. We assume that the focal point lies at geodesic distance s_0. By definition 2.10 the tangent length of the offset curve $\boldsymbol{\alpha}_{s_0}(t)$ at t_0 equals zero, i.e. $y(s_0,t_0) = 0$. Since y satisfies the Jacobi equation (13) and is not identical to zero (otherwise the progenitor curve would be singular at parameter t_0), we have

$$y(s_0,t_0) \;=\; 0, \quad \frac{\partial}{\partial s} y(s_0,t_0) \;=\; y'_{t_0}(s_0) \;\neq\; 0$$

for every focal point of the progenitor curve. Using the implicit function theorem, the equation $y(s(t),t) = 0$ can be solved to give a differentiable real-valued function $s(t)$ in a neighbourhood $J \subset I$ of t_0. Moreover, $\frac{\partial}{\partial s} y(s(t),t) \neq 0$ holds for all $t \in J$. Applying the chain rule yields

$$\frac{d}{dt}\left(y(s(t),t)\right) \;=\; \frac{\partial}{\partial s} y(s(t),t)\, s'(t) + \frac{\partial}{\partial t} y(s(t),t).$$

Hence the ordinary differential equation

$$s'(t) \;=\; -\frac{\frac{\partial}{\partial t} y(s(t),t)}{\frac{\partial}{\partial s} y(s(t),t)} \tag{26}$$

is valid for all $t \in J$. The partial derivatives in equation (26) can be computed using the Jacobi equation (13) and equation (23) from above.

The concept outlined appears to be quite promising in order to trace the focal curve of a given progenitor curve $\boldsymbol{\alpha}(t)$ by interpreting the tangent vector $(s'(t), 1)$ in the (s,t) parameter space. Details on the derivation of differential equation (26) along with a description of future numerical experiences will appear elsewhere.

6 Conclusions

In this paper, we introduce a method for tracing the medial curve of two given analytic border curves on an arbitrary regular surface. First numerical experiments convinced us that the algorithm yields good results. In all the examples considered so far an accuracy of about 10^{-10} appears to be feasable. Future works will deal with numerical error estimations and consequences on feasible geometric accuracies.

As stated above, the method is useful in the planar case too. Here geodesic lines can be replaced by the progenitor curve's normals and the length of their offset curves can be computed explicitly. Thus the computational cost of the algorithm can be reduced enormously. On the other hand, many of the phenomena (e.g. focal points, „medial points“ that are not equidistantial to the border curves) presented in this paper may even occur in the Euclidean case. For this reason (and of course, because there are many obvious applications in the planar case) we currently work on this as a special case.

Provided the algorithm is initialized with a medial point which is truly equidistant to the border curves (i.e. the global geodesic distance from this point to both border curves equals the considered one), then usually (exceptions are rare but possible) the equidistantial set is traced with the method described above in some neighbourhood of the starting point. Therefore the considered method can be a useful tool to trace the equidistantial set locally. Obviously some global problems remain to be solved before an algorithm for computing equidistantial sets or medial axis can be formulated which is based on our method.

We have shown that the considered differential equation becomes singular if and only if the medial curve passes a focal point of one of the border curves. It is crucial for global distance computations that a minimal geodesic segment will not be distance minimal after passing its related focal point[1]. These are some of the reasons why the study of focal curves is of certain interest. We sketched out how the Jacobi vector field and the computation of its variation in length can be used for tracing the focal curve as well. Although the mathematical formulation is valid, numerical experiments are needed in the future to prove the concepts outlined here to be useful from the computational point of view.

Bibliography

[BJA91] J. W. Brandt, A. K. Jain, and V. R. Algazi. Medial Axis Representation and Encoding of Scanned Documents. *Journal of Visual Communication and Image Representation*, 2(2):151–165, June 1991.

[BL73] W. Blaschke and K. Leichtweiß. *Elementare Differentialgeometrie.* Springer-Verlag, Berlin, 1973.

[dC76] P. M. do Carmo. *Differential Geometry of Curves and Surfaces.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1976.

[DH93] D. Dutta and C. M. Hoffmann. On the Skeleton of Simple CSG Objects. *Journal of Mechanical Design, ASME Transactions*, 115(1):87–94, March 1993.

[Gur89] H. N. Gursoy. *Shape Interrogation by Medial Axis Transform for Automated Analysis.* PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, November 1989.

[1] Note that a geodesic segment emanating orthogonally from a curve may have lost its distance minimal property before reaching a related focal point because it intersected some other minimal geodesics.

[HV94] C. M. Hoffmann and P. J. Vermeer. Validity Determination for MAT Surface Representation. In *Sixth IMA Conference on the Mathematics of Surfaces*, IMA Conference Series. Oxford University Press, September 1994.

[Mae96] T. Maekawa. Computation of Shortest Paths on Parametric Free-Form Surfaces. *Journal of Mechanical Design, ASME Transactions*, 1996. To appear.

[NAG89] Numerical Algorithms Group, Oxford, England. *NAG Fortran Library Manual*, Mark 13 edition, 1989.

[PB89] N. M. Patrikalakis and L. Bardis. Offsets of Curves on Rational B-spline Surfaces. *Engineering with Computers*, 5:39–46, 1989.

[PG90] N. M. Patrikalakis and H. N. Gursoy. Shape Interrogation by Medial Axis Transform. In B. Ravani, editor, *Proceedings of the 16th ASME Design Automation Conference: Advances in Design Automation, Computer Aided and Computational Design, Vol. I*, pages 77–88, Chicago, IL, September 1990. New York: ASME.

[She95] E. C. Sherbrooke. *3-D Shape Interrogation by Medial Axis Transform.* PhD thesis, Massachusetts Institute of Technology, Cambridge, Massachusetts, April 1995.

[SPB95] E. C. Sherbrooke, N. M. Patrikalakis, and E. Brisson. Computation of Medial Axis Transforms of 3-D Polyhedra. In C. Hoffmann and J. Rossignac, editors, *Proceedings of the Third ACM Solid Modeling Conference, May 1995, Salt Lake City, Utah*, pages 187–199, New York, 1995. ACM.

[SPW95] E. C. Sherbrooke, N. M. Patrikalakis, and F.-E. Wolter. Differential and Topological Properties of Medial Axes. *Computer Vision, Graphics and Image Processing*, 1995. To appear.

[Sto69] J. J. Stoker. *Differential Geometry.* John Wiley & Sons, Inc., New York, 1969.

[Wal93] W. Walter. *Gewöhnliche Differentialgleichungen.* Springer-Verlag, Berlin, 1993.

[Wol85] F.-E. Wolter. *Cut Loci in Bordered and Unbordered Riemannian Manifolds.* PhD thesis, Technical University of Berlin, Department of Mathematics, December 1985.

[Wol89] F.-E. Wolter. Distance Computations for Curves and Surfaces. Mini-Symposium on Engineering Geometry at SIAM Conference on Geometric Design, Tempe, Arizona, 1989.

[Wol92] F.-E. Wolter. Cut Locus and Medial Axis in Global Shape Interrogation and Representation. Memorandum 92-2, Cambridge MA: MIT Ocean Engineering Design Laboratory, January 1992.

[Wol95] F.-E. Wolter. Geodesic Offsets, Computations and Applications. International Symposium on Freeform Curves and Surfaces 95, Mathematical Research Institute Oberwolfach, Germany, 1995.

[Wol96] F.-E. Wolter. Geodesic Offsets and Medial Curves. Dagstuhl-Seminar on Geometric Modeling at the International Conference and Research Center for Computer Science, Dagstuhl Castle, Germany, 1996.